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LETTER TO THE EDITOR

Chaotic analytic zero points: exact statistics for those of a random spin state

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Abstract. A natural statistical ensemble of $2J$ points on the unit sphere can be associated, via the Majorana representation, with a random quantum state of spin J, and an exact expression is obtained here for the general k point correlation function ρ_k in this ensemble. The pair correlation ρ_2 in the large-J limit takes the relatively simple form $(J/2\pi)^2 g(\sqrt{J/2}\theta)$ where $g(r) = [(\sinh^2 r^2 + r^4)\cosh r^2 - 2r^2 \sinh r^2]/\sinh^3 r^2$ and θ is the angular separation of the pair of points on the sphere. It appears (from the numerical work of others) that, in this limit, these statistics are typical of the zero points of analytic functions associated with chaotic quantum dynamical systems.

A general state $\Sigma a_m|J, m\rangle$ of spin J is described in the Majorana representation [1] (except for its overall phase) by $2J$ points on the unit sphere. Their $2J$ unit vectors $\boldsymbol{r}_i$ represent (except for phase) the $2J$ spin-$\frac{1}{2}$ states $|\boldsymbol{r}_i\rangle$, whose symmetrized (and normalized) product is $\Sigma a_m|J, m\rangle$. This representation has the advantage of democracy: there is no longer any priviliged axis inherent in the $|J, m\rangle$ representation. If the spin-J state is rotated in any way, the sphere is rotated in the same way. The relationship between the coefficients a_m and the unit vectors $\boldsymbol{r}_i$ is expressed through the stereographic projection from the south pole of the sphere (defined as the negative m direction) onto its equatorial plane. If the stereographic projections of the unit vectors are denoted by the $2J$ complex numbers $(z_1, z_2, \ldots, z_{2J})$, then these are the zeros of a polynomial $f(z)$ whose coefficients are a_m multiplied by a numerical factor:

$$f(z) = z^J \sum_{-J}^{J} (-1)^{m-J} \sqrt{\frac{(2J)!}{(J+m)!(J-m)!}} a_m z^m \tag{1}$$

$$= a_J (z - z_1)(z - z_2) \cdots (z - z_{2J}) \tag{2}$$

(the z subscripts are not related to the a subscripts). First discovered by Majorana [1] in 1932, this representation has been rediscovered several times since then [2–4]. It has recently been made known to a wider audience by Penrose [5, 6].

Zero points of analytic functions have found application [4, 7, 8] as a striking geometrical characterization of the quantum wavefunction of a dynamical system. The analytic function (the 'Bargmann' function) is generated by taking the inner product of the state of the system with a coherent state centred at z. States of classically integrable systems tend to have zeros lying along one-dimensional curves whereas classically chaotic systems have zeros scattered about in two dimensions. For a spin system the analytic function is found [4] to be exactly the Majorana polynomial, and a typical spin state (i.e. a typical set of coefficients a) has a scatter of zero points looking much like that of a chaotic wavefunction of a dynamical system. The question naturally arises 'what are the statistics of the Majorana zeros for a spin state picked at random?' This is asked and answered here for its own sake, though these statistics might then be hoped to have wider signficance, and it appears from work of others mentioned at the end of this paper that this is so.

The ensemble of configurations of the $2J$ points on the unit sphere is to be generated by a state of spin J chosen uniformly randomly on the unit $4J+1$ sphere $\Sigma|a_m|^2 = 1$ in Hilbert space. In fact, since the configuration does not depend on the state being well normalized (only the ratios of the as matter), one can choose the as from any spherically symmetric distribution in Hilbert space, conveniently a Gaussian with $\langle a_m a_n^* \rangle = \delta_{mn}$, and $\langle a_m a_n \rangle = 0$. A study of general random polynomials with various Gaussian distributions of coefficients has been carried out by Bogomolny, Bohigas and Leboeuf (BBL1) [9]. In particular, they report that the average density of zeros for equation (1) with this distribution gives a uniform density ρ_1 of zeros on the unit sphere. This is as one would hope with the 'random-state' interpretation just mentioned. (Their calculation is to appear (BBL2) [10].)

The joint probability $\rho_k(z_1, z_2, \ldots, z_k)\, \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 \ldots \mathrm{d}^2 z_k$ that k of the $2J$ points lie in $\mathrm{d}^2 z_1, \mathrm{d}^2 z_2, \ldots, \mathrm{d}^2 z_k$ in the complex plane will be obtained. From that, the joint probability $\rho_k(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_k)\, \mathrm{d}^2\boldsymbol{r}_1\, \mathrm{d}^2\boldsymbol{r}_2 \ldots \mathrm{d}^2\boldsymbol{r}_k$ of the points on the unit sphere follows merely by multiplying by the stereographic Jacobian factor $(1+z_1 z_1^*)^2 \ldots (1+z_k z_k^*)^2/4^k$. Either of these two correlation functions ρ_k, when integrated over its whole domain, yields $2J!/(2J-k)!$.

The system is somewhat similar to the two-dimensional one-component plasma (at $1/kT = 2$) for which exact statistics were obtained by Jancovici [11] by interpreting, in a physical context, random matrix results of Ginibre [12]. The analogous results for a sphere rather than a plane were obtained by Caillol [13]. The starting point for the plasma analysis is the full joint probability distribution ρ_{2J}, or partition function, governed by the pairwise Coulomb potential. For the present Majorana system, although ρ_{2J} can, and will in the conclusion, be written down in the analogous form, it is more complicated (it does not correspond to a purely pairwise interaction). Instead, following BBL1, an approach based on equations (1) and (2) and the statistics of the as is to be used, albeit differently implemented.

With $f_j \equiv f(z_j)$ and $f_j' \equiv \mathrm{d}f(z_j)/\mathrm{d}z_j$, the starting point is the joint probability $P(f_1, \ldots, f_k, f_1', \ldots, f_k')\, \mathrm{d}^2 f_1 \ldots \mathrm{d} f_k\, \mathrm{d}^2 f_1' \ldots \mathrm{d}^2 f_k'$. This is Gaussian since the $f(z)$ is linearly related to the coefficients a_m, and is thus fully determined by its $2k \times 2k$ (Hermitian) correlation matrix M made up of four $k \times k$ submatrices:

$$P(f_1, \ldots, f_k, f_1', \ldots, f_k') = \pi^{-k} \det M^{-1} \exp[-(f_1^*, \ldots, f_k^*, f_1'^*, \ldots, f_k'^*) M^{-1} \times (f_1, \ldots, f_k, f_1', \ldots, f_k')] \tag{3}$$

with

$$M = \begin{bmatrix} \langle f_1 f_1^* \rangle & \langle f_1 f_2^* \rangle & \cdots & \langle f_1 f_1'^* \rangle & \langle f_1 f_2'^* \rangle & \cdots \\ \langle f_2 f_1^* \rangle & \langle f_2 f_2^* \rangle & \cdots & \langle f_2 f_1'^* \rangle & \langle f_2 f_2'^* \rangle & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \langle f_1' f_1^* \rangle & \langle f_1' f_2^* \rangle & \cdots & \langle f_1' f_1'^* \rangle & \langle f_1' f_2'^* \rangle & \cdots \\ \langle f_2' f_1^* \rangle & \langle f_2' f_2^* \rangle & \cdots & \langle f_2' f_1'^* \rangle & \langle f_2' f_2'^* \rangle & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{bmatrix} \equiv \begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix}. \tag{4}$$

The k point correlation in a spin-J system, can now be calculated. The result (6) is valid for all k values, that is $1 \leqslant k \leqslant 2J$ (see the note following (6)):

$$\begin{aligned}
\rho_k(z_1, \ldots, z_k) &= \int \ldots \int P(0, \ldots, 0, f'_1, \ldots, f'_k)|f'_1 \ldots f'_k|^2 \, \mathrm{d}^2 f_1 \ldots \mathrm{d}^2 f_k \, \mathrm{d}^2 f'_1 \ldots \mathrm{d}^2 f'_k \\
&= \pi^{-k} \det M^{-1} \int \ldots \int \exp[-(0, \ldots, 0, f'^*_1, \ldots, f'^*_k) M^{-1} \\
&\quad \times (0, \ldots, 0, f'_1, \ldots, f'_k)] |f'_1 \ldots f'_k|^2 \, \mathrm{d}^2 f_1 \ldots \mathrm{d}^2 f_k \, \mathrm{d}^2 f'_1 \ldots \mathrm{d}^2 f'_k \\
&= \pi^{-k} \det M^{-1} (-1)^k \text{ coeff of } [\mu_1 \ldots \mu_k \mu^*_1 \ldots \mu^*_k] \\
&\quad \text{in } \int \ldots \int \exp[-(0, \ldots, 0, f'^*_1, \ldots, f'^*_k) M^{-1} (0, \ldots, 0, f'_1, \ldots, f'_k)] \\
&\quad \times \exp[\mathrm{i}(\mu^*_1 f'_1 + \cdots + \mu^*_k f'_k + \mu_1 f'^*_1 \ldots \mu_k f'^*_k) \, \mathrm{d}^2 f_1 \ldots \mathrm{d}^2 f_k \, \mathrm{d}^2 f'_1 \ldots \mathrm{d}^2 f'_k \\
&= \pi^{-k} \det M^{-1} \det N (-1)^k \text{ coeff of } [\mu_1 \ldots \mu_k \mu^*_1 \ldots \mu^*_k] \\
&\quad \text{in } \exp[-(\mu^*_1, \ldots, \mu^*_k) N (\mu_1, \ldots, \mu_k)] \\
&= \pi^{-k} \det M^{-1} \det N \text{ per } N
\end{aligned} \tag{5}$$

where $N \equiv C - B^\dagger A^{-1} B$ is the $k \times k$ (Hermitian) matrix whose inverse is the lower right submatrix of the inverse matrix M^{-1}. The notation 'per N' means the 'permanent' of N (per $N \equiv \Sigma \Pi N_{jPj}$ whereas $\det N = \Sigma \Pi (-1)^P N_{jPj}$, where the product is over all i from 1 to k, and the sum is all over $k!$ permutations P, with signature denoted $(-1)^P$). The final product can be simplified by Jacobi's theorem [14], $\det M^{-1} \det N = 1/\det A$ so that, finally,

$$\rho_k(z_1, \ldots, z_k) = \pi^{-k} \text{ per } [C - B^\dagger A^{-1} B] / \det A \,. \tag{6}$$

This is the result. It should be noted that for the high k values $2J + 1 < 2k \leqslant 4J$ the intermediate equations involve cancelling singularities because there are then more variables $f_1, \ldots, f_k, f'_1, \ldots, f'_k$ than coefficients $a_{-J}, \ldots, a_J$ generating them. Both $\det M$ and $\det N$ are then zero, but only their ratio $\det A$, which is well defined, appears in the result which thus is valid for all k.

For the stated distribution of coefficients a we have

$$\langle f_i f^*_j \rangle = \langle a_{-J} a^*_{-J} \rangle + \langle a_{-J+1} a^*_{-J+1} \rangle (2J)(z_i z^*_j)^1 + \cdots \langle a_J a^*_J \rangle (z_i z^*_j)^{2J} = (1 + z_i z^*_j)^{2J} \tag{7}$$

$$\langle f_i f'^*_j \rangle = [\partial/\partial z^*_j](1 + z_i z^*_j)^{2J} = 2J z_i (1 + z_i z^*_j)^{2J-1} \tag{8}$$

$$\langle f'_i f^*_j \rangle = [\partial/\partial z_i](1 + z_i z^*_j)^{2J} = 2J z^*_j 1 + z_i z^*_j)^{2J-1} \tag{9}$$

$$\langle f'_i f'^*_j \rangle = [\partial/\partial z_i \partial/\partial z^*_j](1 + z_i z^*_j)^{2J} = 2J(1 + 2J z_i z^*_j)(1 + z_i z^*_j)^{2J-2} \,. \tag{10}$$

This completes the description of the general k point correlation. The limit as $J \to \infty$ is obtained in each equation by replacing the brackets $(1 + \bullet)^{2J}$ by $\exp(2J\bullet)$ and (in the last three) differentiating it. The results are, respectively,

$$\exp[2J z_i z^*_j] \qquad 2J z_i \exp[2J z_i z^*_j] \qquad 2J z^*_j \exp[2J z_i z^*_j] \qquad 2J(1 + 2J z_i z^*_j) \exp[2J z_i z^*_j] \,.$$

The correlation functions thus shrink in width like $J^{-1/2}$ as $J \to \infty$ as should be expected.

The one-point correlation or density $\rho_1(z)$ is $2J/\pi(1 + zz^*)^2$, giving $\rho_1(\boldsymbol{r}) = 2J/4\pi$, the uniform distribution (BBL1 [6], BBL2 [7]).

The pair correlation ρ_2 is also short enough to write out. It depends, of course, only on the separation of the points on the sphere, so it suffices to pick any two points, say 0 and z on the plane:

$$\rho_2(0, z) = [(2J(a-1) - b^2)(d(a-1) - c^2) + (2J(a-1) - bc)^2]/\pi^2(a-1)^3 \tag{11}$$

where $a = (1+zz^*)^{2J}$, $b = 2J|z|$, $c = 2J|z|(1+zz^*)^{2J-1}$, $d = 2J(1+2Jzz^*)(1+zz^*)^{2J-2}$. To convert this to $\rho_2(\boldsymbol{r}_1, \boldsymbol{r}_2)$ for points of angular separation θ on the sphere one multiplies $\rho_2(0, z)$ by the Jacobian factor $(\frac{1}{4})(1 + zz^*)^2/4$ and substitutes $z = \tan(\theta/2)$.

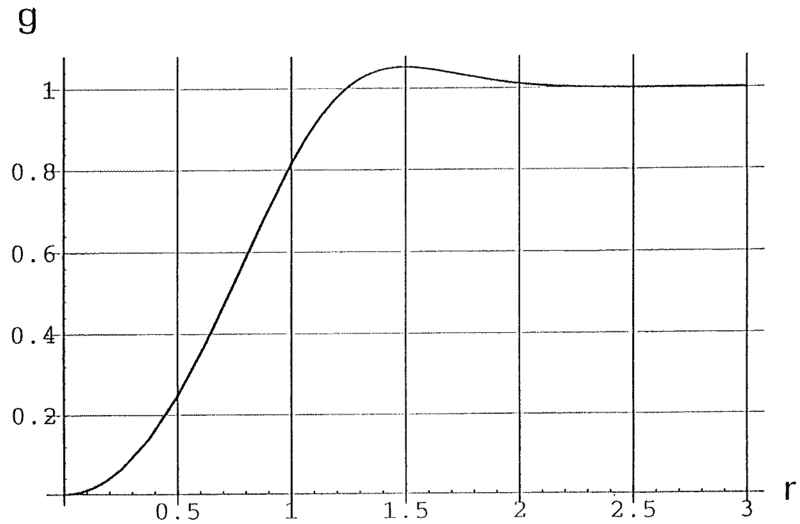

Figure 1. The function $g(r) = [(\sinh^2 r^2 + r^4)\cosh r^2 - 2r^2 \sinh r^2]/\sinh^3 r^2$. The $J \to \infty$ limit of the pair correlation function ρ_2 is $(J/2\pi)^2 g(\sqrt{J/2}\theta)$ where θ is the angular separation of the pair of points on the sphere.

The $J \to \infty$ limit of $\rho_2(0, z)$ is $(2J/\pi)^2 g(\sqrt{2Jzz^*})$ where

$$g(r) = [(\sinh^2 r^2 + r^4)\cosh r^2 - 2r^2 \sinh r^2]/\sinh^3 r^2 \qquad (12)$$

which has the quadratic repulsion $g(r) \sim r^2$ near the origin, and a single small hump as shown in the figure. On the sphere the Jacobian factor $1/4^2$ gives the limit of $\rho_2(\boldsymbol{r}_1, \boldsymbol{r}_2)$ as $(J/2\pi)^2 g(\sqrt{2J}\tan(\theta/2))$ which is equivalent, in the limit $J \to \infty$, to $(J/2\pi)^2 g(\sqrt{J/2}\theta)$, because in this limit, unless θ is small, $g(r) \to 1$, with or without the tan.

In conclusion, mention should be made of a simple alternative formula for ρ_{2J}, that is, the probability distribution for all the points, which is easily derived from the analysis in BBL1 [9] (and indeed to appear in BBL2 [10]). The key relation is the 'well known' one cited in BBL1 between the roots of a polynomial and its coefficients:

$$d^2 a_{-J} \ldots d^2 a_J = \prod_{i=1}^{2J} \prod_{j=1}^{i-1} |z_i - z_j|^2 \, d^2 z_1 \ldots d^2 z_{2J} |a_J|^{4J} \, d^2 a_J \, . \qquad (13)$$

Using this, together with the Gaussian probability distribution for the as (with zs substituted for the as from (1) and (2)), one obtains

$$\rho_{2J}(z_1, z_2, \ldots, z_k) = \frac{2J!}{\pi^{2J} \prod_{j=1}^{2J} 2J!/(2J-j)!j!} \frac{\prod_{i=1}^{2J} \prod_{j=1}^{i-1} |z_i - z_j|^2}{\left\{ \sum_{\text{Perms P}} \prod_{i=1}^{2J} (1 + z_i z_{Pi}^*) \right\}^{2J+1}} \, . \qquad (14)$$

This must be equivalent to the general formula (6) in the case $k = 2J$. The numerator is essentially the 2D one-component plasma form of ρ_{2J} mentioned earlier, and is responsible for the quadratic form of ρ_2 near the origin. The denominator evidently does not represent a pairwise interaction.

It is interesting to convert the ρ_{2J} of (14) to the sphere geometry. The necessary relation is $\langle \boldsymbol{r}_i|\boldsymbol{r}_j\rangle = (1+z_i z_j^*)/\sqrt{1+z_i z_i^*}\sqrt{1+z_j z_j^*}$ (having taken an immaterial, but convenient choice of phase zeros for the states). This is to be used on the product of (14) and the stereographic Jacobian factor $(1+z_1 z_1^*)^2 \ldots (1+z_{2J} z_{2J}^*)^2/4^{2J}$. If this factor is denoted q^2 and written as $q^{-(2J-1)}/q^{-(2J+1)}$, then multiplying (14) by this ratio, the numerators combine, and the denominators combine to give

$$\rho_{2J}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_k) = \frac{2J!}{(4\pi)^{2J} \prod_{j=1}^{2J} 2J!/(2J-j)!j!} \frac{\prod_{i=1}^{2J}\prod_{j=1}^{i-1}\left(1-|\langle \boldsymbol{r}_i|\boldsymbol{r}_j\rangle|^2\right)}{\left\{\sum_{\mathrm{PermsP}}\prod_{i=1}^{2J}\langle \boldsymbol{r}_i|\boldsymbol{r}_{Pi}\rangle\right\}^{2J+1}}. \tag{15}$$

This can be interpreted in words as follows. The quantity $|\langle \boldsymbol{r}_i|\boldsymbol{r}_j\rangle|^2$ equals $(1+\boldsymbol{r}_i\cdot\boldsymbol{r}_j)/2$, so that the numerator is the product, over all distinct pairs, of terms $(1-\boldsymbol{r}_i\cdot\boldsymbol{r}_j)/2$. This term is the square of half the chord length between points i and j. The braces in the denominator contain the sum over all permutations P of the product of terms like $\langle \boldsymbol{r}_i|\boldsymbol{r}_j\rangle\langle \boldsymbol{r}_j \boldsymbol{r}_k\rangle \ldots \langle \boldsymbol{r}_l|\boldsymbol{r}_i\rangle$, one from each cycle in P. This term has a magnitude equal to the product of all the radii of the chord midpoints, and a phase equal to half the (signed) area, or solid angle, of the spherical polygon of geodesics formed by the cycle of points on the sphere.

Finally, as indicated earlier, the statistics obtained above appear to apply beyond the static, but random, spin system considered here. Leboeuf and Shukla [15] have, I learn, examined the pair correlation ρ_2 numerically, and compared the results with the correlation of complex zeros of the analytic functions ('Bargmann' functions) associated with the wavefunctions of chaotic dynamical systems. The systems they studied are the kicked top and the standard map on the torus, and another system examined by Prosen [8] yields similar results (personal communication). The agreement obtained lends support to the universality that they suggest applies to the statistics of zeros of 'chaotic' analytic functions. Systems with time reversal symmetry would require a modified analysis and this has now been supplied by Prosen [16].

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